M.Sc. - Mathematics
I Semester End Examination - May 2022
Ordinary Differential Equations

Course Code: MM104T **QP Code: 11004**
Time: 3 hours **Total Marks: 70**

Instructions: 1) **All** questions carry **equal** marks.
2) Answer **any five** full questions.

1 a) State and prove Liouville's theorem.

b) If $cos^3 x$ is one of the solutions of $y'' + 2\,tan\,x\,y' + y = 0$ determine the second solution.

c) Define fundamental set. Show that the solution set of $\frac{d^4y}{dx^4} + 6\frac{d^3y}{dx^3} + 11\frac{d^2y}{dx^2} + 6\frac{dy}{dx} = 0$ is a fundamental set. (6+4+4)

2 a) State and prove Strum's comparison theorem.

b) Solve by the method of variation of parameters of the equation $x^2y'' - xy' - 3y = x^3$.

c) Define self-adjoint differential equation. Find the adjoint of the differential equation

$x^4y''' + 3x^3y'' + 5x^2y' + 12xy = 0$. (7+4+3)

3 a) Establish the eigen function expansion formula for an eigen value problem. Expand x in terms of orthonormal eigen function of the eigen value problem $y'' + \lambda y = 0, y(0) = 0,$ $y(\pi) = 0.$

b) Construct the Greens function for $y'' + \lambda y = x, y(0) = 0 = y(1)$. (9+5)

4 a) Prove that the eigenvalues of a self-adjoint eigenvalue problem are real.

b) For the eigenvalue problem $y'' + \lambda y = 0$ in $[0,1]$, show that the non-zero eigenvalue satisfies the equation $tan\sqrt{\lambda} = \sqrt{\lambda}$ under the condition $y(0) = 0,\ y(1) - y'(1) = 0.$

c) Show that $\frac{dy}{dx} = x^2 + y, \quad y(0) = 1$ with $|x| \le 1, |y - 1| \le 1$ has a unique solution in $|x| \le \frac{1}{3}$. (4+5+5)

5 a) Find the power series solution of $(1-x^2)y'' + 2xy' + p(p+1)y = 0$ about an ordinary point.

b) Find the Frobenius solution of $xy'' + 2(1-x)y' + (x-2)y = 0$. (6+8)

6 a) Show that Laguerre polynomials are orthogonal over $(0,\infty)$.

b) Prove that $T_{n+1}(x) - 2xT_n(x) + T_{n-1}(x) = 0, \quad n \geq 1$.

c) Obtain the general solution of $(1-x^2)y'' - xy' + \alpha^2 y = 0$ where α is a constant.

(6+4+4)

7 a) Find the fundamental matrix and general solution of $\frac{d\vec{X}}{dt} = \begin{bmatrix} 4 & 3 & 1 \\ -4 & -4 & -2 \\ 8 & 12 & 6 \end{bmatrix} \vec{X}$;

$\vec{X} = [x, y, z]^T$.

b) Determine the solution of $\frac{dx}{dt} = -3y + e^{5t}$, $\frac{dy}{dt} = 2x + 4$, with $x(0) = 0, y(0) = 0$.

(7+7)

8 a) Determine the nature and stability of the critical point for the system:

i. $x_1' = x_1 - x_2; \quad x_2' = x_1^2 + x_2^2 - 2$.
ii. $x_1' = x_1 - 2x_2 - 1; \quad x_2' = 2x_1 - 3x_2 - 3$.

b) Using the Liapunov function, determine the stability of the critical point (0,0) of the system

$\frac{dx}{dt} = -x^5 - y^3; \qquad \frac{dy}{dt} = 3x^3 - y^3$. (8+6)

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